

EVALUATING DEFINITE INTEGRALS

If the definite integral represents an area between a curve and the x -axis, and if you can find the area by recognizing the shape of the region, then you can evaluate the definite integral. Those are big "ifs." Most of the time the definite integral will not be set up to find an area, let alone a simple area. This chapter will consider a much more efficient way: the Fundamental Theorem of Calculus. First, however, is an example illustrating another technique. This approach may be done in a precalculus course as a find the area between the graph and the axis problem without mentioning Riemann Sum or definite integral.

Areas Without "Calculus"

Evaluate the definite integral $\int_0^1 (x^2 + x + 1)dx$ by writing it as the limit of a Riemann Sum and finding that limit.

The interval is $[0, 1]$ and the function is $f(x) = x^2 + x + 1$. The regularly spaced partition points are at $x = 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k}{n}, \dots, \frac{n-1}{n}, 1$. Forming the limit of the right-hand Riemann Sum gives:

$$\int_0^1 (x^2 + x + 1)dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\left(\frac{k}{n} \right)^2 + \frac{k}{n} + 1 \right) \left(\frac{1-0}{n} \right)$$

Next expand and simplify (k is the variable, n is constant):

$$\lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \left(\frac{k}{n} \right)^2 \left(\frac{1}{n} \right) + \sum_{k=1}^n \left(\frac{k}{n} \right) \left(\frac{1}{n} \right) + \sum_{k=1}^n (1) \left(\frac{1}{n} \right) \right] = \lim_{x \rightarrow \infty} \left[\frac{1}{n^3} \sum_{k=1}^n k^2 + \frac{1}{n^2} \sum_{k=1}^n k + \frac{1}{n} \sum_{k=1}^n 1 \right]$$

There are simple formulas for these three sums (usually proved by mathematical induction in precalculus). This results in the limit of the sum of three rational expressions whose numerators are of the same degree as their denominators. The limits are easily found.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left[\frac{1}{n^3} \sum_{k=1}^n k^2 + \frac{1}{n^2} \sum_{k=1}^n k + \frac{1}{n} \sum_{k=1}^n 1 \right] &= \lim_{x \rightarrow \infty} \left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{2n^2} + \frac{n}{n} \right] \\ &= \frac{2}{6} + \frac{1}{2} + 1 \\ &= \frac{11}{6} \end{aligned}$$

$$\text{So } \int_0^1 (x^2 + x + 1) dx = \frac{11}{6}.$$

Even this method is time consuming, cumbersome, and only applicable to polynomials. The next discussion will lead to a very important result.

Recognizing the Shape

Some Riemann sums may be found by using the shape of the region, for example $\int_{-3}^3 \sqrt{9-x^2} dx = \frac{9}{2}\pi$ since the region is a semicircle with a radius of 3. There are very often integration questions on the AP Exams that give students the graph of a function (with no equation) and ask students to evaluate its definite integral. Students are expected to find the area of the simple shapes based on the graph. (See for example 1999 AB 5 / BC 5 or 2003 AB 4/BC 4 among others.)

Many of the properties of integrals (see Chapter 16) can be seen based on the location and shape of the region between the graph and the x -axis and the location of a and b . So far we've considered functions that were non-negative and monotonic over the interval. Neither of these restrictions is necessary.

If f is negative over the entire interval then the Riemann sum gives the negative of the area of the region between the x -axis (on top) and the curve. This is because in each term the function values are negative. If f is both positive and negative on the interval, the Riemann sum gives the algebraic sum of the "positive" area and the "negative" area. The sum may be positive, negative or zero. Example (with no knowledge of antiderivatives necessary): $\int_0^{2\pi} \sin(x) dx = 0$ by the symmetry of the graph.

Integration is done from left to right. However, if $a > b$, then $\Delta x = \frac{b-a}{n} < 0$ and $\int_a^b f(x) dx = -\int_b^a f(x) dx$. Even this idea may be tested without equations. In 2002 AB 4/ BC 4 the graph of $y = f(t)$ was given without an equation and a function g was defined as $g(x) = \int_0^x f(t) dt$. Students were asked for the value of $g(-1)$ requiring integration from *right* to *left*. The numerical value was found by finding the opposite of the area of the region (a triangle).

Finding the Net Change

Suppose there is a continuous function that runs from the point $(a, f(a))$ on the left to the point $(b, f(b))$ on the right. What is the net change in the y -values of this function from one end to the other? The answer is easy enough: $f(b) - f(a)$. Keep this in mind as we consider another way to find the net change in y .

Partition the interval $[a, b]$ in the usual way, calculate the change in y on each subinterval and sum them up (the changes may be positive, negative, or zero). The

net change is approximately $\sum_{k=1}^n \Delta y_k$.

Unfortunately, this is not a Riemann Sum¹. But nevertheless we would expect its limit as n approaches infinity to be the net change in y .

$$\text{The net change in } y = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta y_k.$$

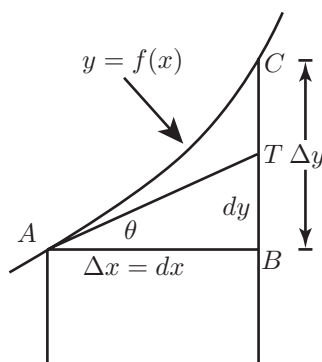


Figure 14.1 A close look at the function on one subinterval.

Consider the part of the graph between any two partition points (See Figure 14.1). Let $AB = \Delta x = dx$. \overline{AT} is tangent to the graph at A and makes an angle of θ with \overline{AB} . The actual change in y , $\Delta y = CB$. The change along the tangent line (for the same change in x) is $dy = TB$. Then as Δx approaches zero, dy approaches Δy . By trigonometry, $dy = (\tan \theta)(dx)$. Since \overline{AT} is tangent at A , $\tan \theta = \text{slope } \overline{AT} = f'(x) = \frac{dy}{dx}$, and $dy = f'(x)dx$.²

$$\text{So } \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta y_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n dy = \lim_{n \rightarrow \infty} \sum_{k=1}^n f'(x_k) \Delta x$$

¹ Actually it is a Riemann sum for the function $f(y) = 1$ on the interval with endpoints of $f(a)$ and $f(b)$. This region is a rectangle with vertical side of $f(b) - f(a)$ and horizontal side of 1. Its value, the area of a rectangle, is $f(b) - f(a)$. But we already knew that.

² Previously you may have defined $dy = f'(x)dx$ and discussed differentials (dy and dx). Their introduction at this point seems natural.

This last limit is a Riemann Sum so the net change in y is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f'(x_k) \Delta x = \int_a^b f'(x) dx$$

And since we already know the net change

$$\int_a^b f'(x) dx = f(b) - f(a)$$

This very important result is known as the **Fundamental Theorem of Calculus**³. It says that if we know the function of which the integrand is the derivative (called its *antiderivative* or its *indefinite integral*), then the definite integral is the difference of the antiderivative evaluated at the upper limit of integration minus the value at the lower limit. Notice also that this equation says that the definite integral of the rate of change is the net amount of change. In fact, this approach started out by finding the amount of change of a function on an interval, *without* assuming it was the accumulated rate of change.

As an example, return to the problem at the beginning of this chapter. Note that if $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$ then $f'(x) = x^2 + x + 1$. That is, $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x$ is an antiderivative of $x^2 + x + 1$. $f(1) = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$ and $f(0) = 0$. Using the Fundamental Theorem of Calculus

$$\int_0^1 (x^2 + x + 1) dx = f(1) - f(0) = \frac{11}{6} - 0 = \frac{11}{6}$$

Students now have a very good reason to want to know how to find antiderivatives. Once an antiderivative is known, evaluating the definite integral involves only evaluating an antiderivative twice and subtracting. The fact that there is more than one antiderivative of a given function turns out not to be a problem.

That so many of the important ideas and concepts relating to integration can be taught and learned *without* knowing how to find antiderivatives is significant. Beginning calculus students get bogged down in the details of finding antiderivatives and often lose the big picture. This is a good reason for doing the big ideas first and separately from the techniques of antidifferentiation. Of course, they will have to learn some of these techniques (Chapter 16) and will use them with the applications (Chapter 17). But first, in the next chapter, we consider more on the concept of accumulation and functions defined by integrals.

³Most, but not all, texts call this the Second Fundamental Theorem of Calculus and the form in Chapter 15, the First.

AP Exam Questions relating to the Fundamental Theorem of Calculus.

The questions that give the graph of the derivative and ask questions about the function including integrals are very common on AP Calculus exams and students should be prepared for them.

1. **1995** AB 6; BC 6
2. **1997** AB multiple-choice 84; BC multiple-choice 82, 89
3. **1998** AB multiple-choice 5, 88; BC multiple-choice 28, 88
4. **1999** AB 5 / BC 5
5. **2000** AB 4;
6. **2001** AB 3 / BC 3
7. **2002** AB 2 - BC 2
8. **2003** Multiple choice: AB 23, 92; BC 18, 27; Free-response AB 4/ BC 4;
Form B AB 5
9. **2004** AB 5
10. **2005** AB 3/BC 3